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AUTHOR Pirie, Susan; Kieren, Thomas
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ABSTRACT

Given the current and widespread practical interest in mathematical understanding, particularly with respect to higher order thinking skills, curriculum reform advocates in many countries cite the need for teaching mathematics with understanding. However, the characterization of understanding in ways that highlight its growth, as well as the identification of pedagogical actions that sponsor understanding, represent continuing problem areas. This paper presents the theoretical basis for a companion document, entitled "The Characteristics of the Growth of Mathematical Understanding." It shows a theory of mathematical understanding that is based on the consideration of understanding as a whole, dynamic, levelled but non-linear process of growth. A model for the theory is represented by eight embedded rings, each representing a level of understanding activity attainable for any particular topic by any specific person. The levels are illustrated by tracing the building of an understanding of the concept of fractions. Beginning with the initial cognitive stage, the levels are called: (1) primitive knowing; (2) image making; (3) image having; (4) property noticing; (5) formalizing; (6) observing; (7) structuring; and (8) inventizing. The application of the theory to teaching is discussed. (MDH)

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A Dynamic Theory of Mathematical Understanding - some features and implications

Susan Pirie
University of Oxford

Thomas Kieren
University of Alberta

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Background to the Theory

There is currently much practical interest in mathematical understanding. Curriculum reform advocates in many countries cite the need for teaching mathematics with understanding. Conference proceedings and psychological and artificial intelligence literature all exhibit interest in learning and teaching with understanding. Characterising understanding in a way which highlights its growth, and identifying pedagogical acts which sponsor it, however, represent continuing problems.

There has been a wide variety in the approaches to attempting to capture the essence of the phenomenon. Various categories of understanding, including relational and instrumental, concrete and symbolic, and intuitive and formal, have been proposed (Skemp (1976), Herscovics, and Bergeron (1988), Schroder (1987)). Alternative views of understanding in relation to cognitive obstacles (Sirpinska (1990)) or in terms of mental objects and connections among them (Ohlsson (1988)) have been proposed. In 1988, Pirie called into question the use of categories in characterising the growth of understanding as it can actually be seen by an observer. She observed understanding as a whole dynamic process and not as a single or multi-valued acquisition, nor as a linear combination of knowledge categories.

It was our wish to better describe this growth of mathematical understanding in the children that we observed in classrooms over time that led to the development of the ideas for our theory. It was clear to us that the children we were observing exhibited some understanding of mathematics, and so our question became "What is mathematical understanding?" Our background thinking was further stimulated by the biological theory of cognition in self-referencing systems (Maturana and Varela (1980, 1987) and Tomm (1989)). Over the past three years we have discussed the our developing theory in a variety of forums (see bibliography), including the 1990 AERA meeting: a theory of the growth of mathematical understanding as a whole, dynamic, levelled but non-linear, transcendently recursive process. This theory attempts to elaborate in detail the constructivist definition of understanding as a continuing process of organising one's knowledge structures (von Glasersfeld,1987).

A Model for the Theory

There are eight potential levels within the growth of understanding for a specific person, for any specified topic, and we illustrate these here by tracing through the building of an understanding of the concept of fractions.

The process of coming to understand starts at a level we call primitive knowing. Primitive here does not imply low level mathematics but rather the starting place for growth for any particular mathematical understanding. It is what the observer, the teacher or the researcher, assumes that the person doing the understanding can do initially; in other words it is what is assumed to be what the learner brings to the task 'in her head'. One can, of course, never know what this really is.

Example: If one wanted to observe the growth in understanding of 10 or 11 year old children studying fractions one might assume they could divide up continuous things - such as pizzas - or sets of discrete things - such as sweets - and that they could use at least some fraction words and symbols descriptively.

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At the second level, the learner begins to form images through specific doing. The learner is being asked to make distinctions in her previous abilities and use them perhaps under new circumstances or to new ends. We call this level image making

Example: With fractions children would successfully engage in solving sharing problems and use fraction language to record their actions. For instance: sharing three pizzas between four people and writing or saying "three quarters of a pizza each."

At the next level these singular activities are replaced by a mental picture. It is this image having which frees a person's mathematics from the need for particular actions or examples. This image is a mental object which can now be used in mathematical knowing.

Example: At this level children might say "fractions show the amounts that come from dividing up" or, more interestingly, "different ways of dividing up into fraction pieces can show the same amount."

At the fourth level, property noticing, these images can now be examined for specific or relevant properties. This involves noting distinctions, combinations or connections between images and predicting how these might be achieved.

Example: Now children might notice that equivalent fractions can be generated by multiplying numerator and denominator by the same value, and can record this process by writing for example "2/3 = 4/6 = 6/9 =..." This allows them to create and talk about fractions that might not be in their direct experience for example 144/216 .The feature that distinguishes this level from that of image having is the ability to not only notice that there is such a thing as equivalence, but also to give some explanation of how to create it.

The following level of understanding, formalising, entails thinking consciously about the noticed properties and abstacting common qualities. It is at this level that full mathematical definitions could occur. The person now has a new class-like mental object.

Example: Children at this level can see things in terms of "for all fractions" and know that a/b can represent any fraction, although they might not be using algebraic language. Fractions now have a new status. They have become a set of numbers or possibly just entities, but certainly no longer action-tied. They can be thought of as formal objects unconnected with specific examples.

One is now in a position to observe ones own formal thought structures and consistently organise them. One is aware of being aware and can see the consequences of ones thoughts and not merely of ones actions.

Example: At this level a child might make the observation "there can be no smallest fractional number." This would contrast with an earlier property " I can make a smaller fraction by adding 1 to the denominator. e.g. 1 over 9 is smaller than 1 over 8." Both would contrast with the even more general but concrete image "more parts means smaller parts." This latter remark is also an illustration of the fact that outer level knowing is not necessarily more general knowing, although it might be.

For deeper understanding one must now be able to answer why these formal observations must be true. This explaining calls for an awareness of associations and sequence within ones formal observations and an ability to use logical arguments to establish their interdependence. This might be setting ones thinking within an axiomatic structure and it is at this level that formal proof will occur. We call this level structuring and question whether this level is not, in fact, peculiar to mathematics.

Example: Here addition of fractions is not combining physical quantities - which would be image making. It is not an action oriented consequence of equivalence - which would occur at the property noticing level. It is not even a formalised algorithm. Addition of fractions is simply a logical property following from other logical properties.

The outermost level of the eight in our model we call the level of inventising. We had some trouble before creating this label as we did not wish to imply that children do not "invent" at other levels. We do, however, wish to point to a special, new activity. Within a given topic a person at this level has a full structured understanding. They may therefore be able to break away from the preconceptions which brought about this complete understanding and create new questions which might grow into a totally new concept.

Example: At the structuring level a person can see the rationals as a set of numbers with the form of an ordered pair, a/b. This set of numbers is also seen to be a quotient field. This person might now inventise by asking: "what might numbers with the form of ordered quadruples a/b/c/d be like?" It was just such a question which stimulated Hamilton to think about and finally develop quaternions from having the structured understanding of the complex numbers.

Features of the Theory

We need to stress at this point that we do *not* see the growth of understanding as a monodirectional process. In an effort to convey this visually, we have presented the model as a sequence of nested circles or layers, thus emphasising the feature that each layer contains all previous layers and is embedded in all succeeding layers. We see growth as represented by back and forth movement between levels and it is thus that we characterise understanding as a dynamic and organising process.

Earlier in this paper, we noted that primitive knowing was the background mathematical understanding needed to build an understanding of some particular concept. It is therefore possible that a full or partial understanding of that concept could then, in turn, be observed as the primitive knowing for a new mathematical exploration. The model has a fractal-like appearance: inspection of any particular primitive knowing will reveal the layers of inner knowings.

One of the strengths of mathematics is the ability to operate at a symbolic level without reference to basic concepts and this is reflected in a critical element of our theory, observable in the model, and illustrated by the bold rings. We call these rings the "don't need" boundaries. The first is between Image Making and Image Having. When a person has an image of a mathematical idea, she does not need actions or the specific instances of image making.

Example : "when I have a mental picture for fractions I can stop cutting up pieces of paper"

The next "don't need" boundary occurs between Property Noticing and Formalising. A person who has a formal mathematical idea does not need an image.

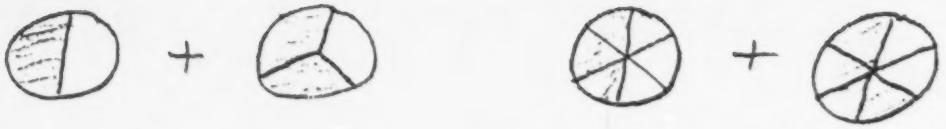
Example : "I can think of fractions as things like a/b with no reference to partitioning".

The third "don't need" boundary occurs between Observing and Structuring. A person with a mathematical structure does not need the meaning brought to it by any of the inner levels.

Example : "I can prove threorems about addition, division etc of ordered pairs without any reference to what a fraction really represents.

The discussion so far focused attention on the levels and indeed they are necessary and structurally important to the theory, but a more crucial feature, alluded to earlier, is that of folding back'. When faced with a problem or question at any level, which is not immediately solvable, one needs to be prompted to fold back to an inner level in order to extend one's current, inadequate understanding. This returned to inner level activity, however, is not identical to the original inner level action - it is now informed and shaped by outer level interests. This inner level action is part of a recursive reconstruction of knowledge, necessary to further build outer level knowing. Let us consider an example of this folding back process, again within the topic of fractions.

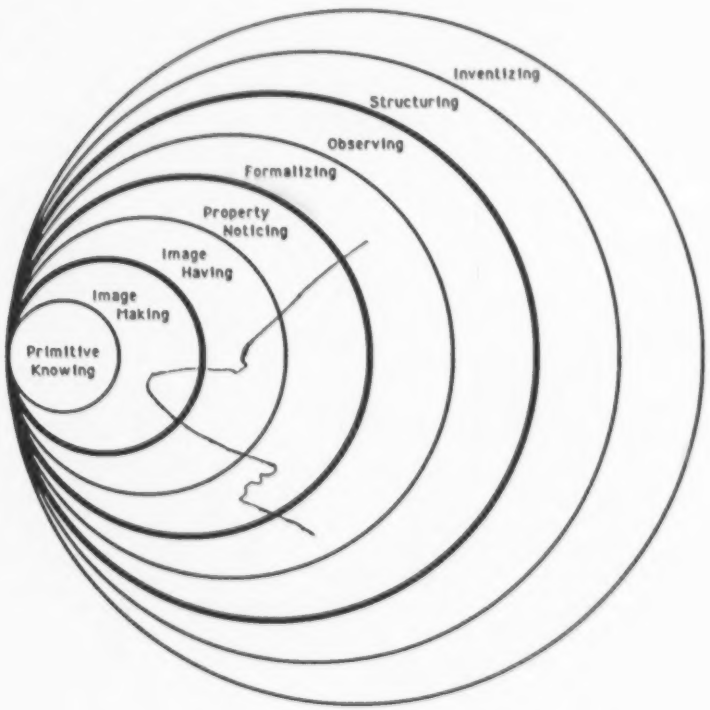
Example: Consider a student who has been formalising and asking" How might one combine fractions?". Clearly one possible route to a solution would be for the teacher to offer the rule: "find a common denominator, and cross multiply to find the numerators and then add the numerators". This would give the student an action to perform but not necessarily any understanding. Alternatively she might be ask "Well what are these things called fractions?" and the response might be "They came from cutting things up - usually pizzas!" and the student might then fold back to drawing pizzas as illustrated below (i), *in order to re-form an image for halves and thirds and combine this with an already noticed property creating equivalent fractions* (ii) with the aim of illuminating the problem of addition.



(i) (ii)

Once the pizzas are all divided into 1/6ths then it may seem sensible to put the 3/6th and 4/6ths together as 7/6ths, or a whole one and an extra 1/6th. After possible further similar calculations the student may then be ready to accommodate the teacher's earlier rule with understanding.

Thus from the level of formalisation the student folded back to image having to make some sense of the operation required at the formalising level. The property of equivalence could then be used for the purpose of creating a meaning for addition of fractions. In fact, the original image became enriched by the idea that one can combine, as well as divide up, fractions. It would have been of no value to simply re-call previous actions. The student needed to re-member and combine existing images to form a new way of looking. This folding back enabled a reconstruction of inner level knowing as a foundation for outer level understanding. The diagram below illustrates the path of growth of understanding that this particular folding back enabled.



Path of growth of understanding

As is obvious from the above example it is not essential for a person to have a complete knowledge of a topic at any particular level in order to move outwards. Indeed quite the opposite can be true. One may move outwards 'at speed' until something triggers the need to fold back, in order to alter or add to ones previous image, property list, or formalisation. Initial understanding can then be narrow and deep. Alternatively one may spend time creating a broad, rich image before moving outwards to seek properties and formalisations. Nevertheless, in this situation there will come a point at which even this image is insufficient for the growth of understanding at an outer level and some folding back will be necessary. This can happen directly to any inner level or through re-tracing the path of growth through the intervening levels. The nature of the path of this folding back, however, cannot be generally prescribed; it is uniquely drawable only for specific examples of growth in understanding such as that exemplified earlier, for the addition of fractions. Each person will have a unique path for any topic, and yet all paths will involve 'folding back to move out' in their actualisation.

A note of caution should be sounded here. Simply asking students to engage in what appears to be high level mathematics is not to be mistaken with providing them with deep understanding. Giving a student a theorem and a proof, and then having the student mimic this proof on closely related questions should not be considered as necessarily providing deep understanding. However, giving the proof to students to work on, could engage them in folding back, in order to make an image of this mathematical proof. This activity could lead to genuine deeper understanding.

The final element of the theory that we wish to mention here is that of the structure within the levels themselves. Each level beyond primitive knowing is composed of a complementarity of acting and expressing and each of these aspects of the understanding growth is necessary to complete a level. Furthermore growth occurs through, at least, first acting then expressing, but more often through to-and-fro movement between these complementary aspects. At any level, acting encompasses all previous understanding, providing continuity with inner levels, and expressing gives distinct substance to that particular level.

Applications of the Theory

With Einstein (quoted in Fine, 1986) we see a theory as: "a self-sharpening tool whose warrants and value in the end rest on this, that they permit the coordination of experience, 'with dividends' [mit vorteil]." So what are the 'dividends' that this theory of the growth of mathematical understanding can offer?

We have used it in a variety of learning environments as a tool to observe the mathematical behaviour of students as they work on a single mathematical task and as they build and organise mathematical knowledge structures over periods of time. The theory has enabled us to comment closely on the levels at which different students are making sense of their mathematical activities and thoughts. (Pirie and Kieren, 1989; Kieren and Pirie [forthcoming]; Pirie and Newman (1990)) Such insight into students' understandings have been used to provide a frame for planning and engaging in mathematics lessons and, in addition, to make observations about curriculum development (Kieren - in collaboration with Pirie - forthcoming).

The scope of this paper allows us to examining only one particular application of the theory in greater detail.

What can a teacher do to help promote growth of understanding? One consequence of the theory is very clear - teachers cannot *give* students understanding. Only the student can build her own understanding. The role of the teacher is to provoke and enable this growth. Our earlier example, of the value of giving a bald algorithm versus encouraging a student to fold back when exploring the addition of fractions, illustrates a key feature of the process. It was the question asked by the teacher which caused the student to fold back to

the image she had of fractions.We see three kinds of 'questioning' as crucial to this task. (questioning is used here to mean more than mere interogation.) We have called these pro-vocative, in-vocative, and validating questions. Pro-vocative questions are those which point a student towards an outer layer, which enable her to move forward in her understanding of some particular piece of mathematics. In-vocative questions are those which cause a student to see the need to fold back to an earlier level in order to progress. Validating questions are those which establish the working of a student within some level, often with the effect of encouraging the expression, verbal or symbolic, of current mathematical actions. The interesting feature of these types of questions is that it is the student and not the teacher who determines the nature of the question.

Example:: Imagine a student who has the image that "fractions show the amounts that come from dividing up whole things". The teacher asks the question " If you had cut a pizza into thirds and you then cut each of those pieces in half what would you have?" Consider now these three possible responses:

(i) "I don't know. I'll have to draw it out and see." For this student the question is acting as an invocative question; she is folding back to image making.

(ii) "I would get 6 pieces altogether" This student is validating the image she has by using it to solve the problem.

(iii) "Each third would make 2 pieces, so there would be three times two pieces altogether, making six pieces. So one third is the same as two sixths". Here the question has provoked the student to explicitly notice the propertyof equivalence. It would be quite possible for a teacher to pose what she hoped would be a pro-vocative question only to find that in fact the current state of the student's understanding caused that student to interpret it in-vocatively, or merely use it to validate current existing understanding.

Summary

The purpose of this paper has been to show a theory of mathematical understanding which is based on the consideration of understanding as a whole, dynamic, levelled but non-linear process of growth. This theory demonstrates understanding to be a constant, consistent organisation of ones knowledge structures: a dynamic process, not an acquisition of categories of knowing.

Its levelled nature has been illustrated through the model of eight embedded rings, each of which represents a level of understanding activity potentially attainable for any particular topic by any specific person. These levels range outwards from the existing knowing that the person brings to the task, through the making and having of an image, to the noticing of properties and formalisation of that image. Reflection on the formalisation leads to observing the thought structures and to their consistent re-organisation. Logical arguments at the next level provide the necessary axiomatic structure to complete the understanding of the topic and leave the knower with the freedom to perhaps mentally alter something within that structure and explore the new field of mathematics thus created. A key tenet of the theory is that outer level knowing does not necessarily mean higher level mathematics. Equally, the converse is also true: high status mathematical topics need to be worked on at the image making level before one can begin to look for an appropriate formalisation or structure.

This theory of growth of understanding has the built-in dynamic of folding back to move out and such growth can be thought of as a continuous path traced back and forth through the levels of knowing It is a non-linear phenomenon which involves folding back to re-member and to re-construct new understanding.

Other crucial features of the model include the notion of "don't need" boundaries to explicate the unique power of mathematics to solve problems in non-image related, symbolic ways. The complementarities within each level provide the link between acting and explaining, and together contain the necessary wholeness of the very nature of understanding.

Our final concern was to offer just one element of the pedagogical implications of the theory: that of the effects of provocative, invocative and validating questions on growth of understanding. The vital aspect of this notion is that it is the student who determines the nature of the question, not the question itself, nor the intentions of the asker.

We'll end with a favourite quotation from Maturana:

"At this point there is either much more to say
or nothing."

When we first enunciated our theory, we were not sure which of these options obtained. We hope now that we have demonstrated that there is indeed much more to say.

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